On the classical capacity of quantum Gaussian channels

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Abstract. The set of quantum Gaussian channels acting on one bosonic mode can be classified according to the action of the group of Gaussian unitaries. We look for bounds on the classical capacity for channels belonging to such a classification. Lower bounds can be efficiently calculated by restricting to Gaussian encodings, for which we provide analytical expressions.

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1. Introduction

Gaussian processes are ubiquitous in physics, mathematics and information theory [1]. In information theory, Gaussian channels are used to model noisy communication lines where the noise is described as a Gaussian process. A central problem in this field is to determine the capacity of the communication line, that is, the maximum rate at which information can be reliably transmitted via the noisy channel, with asymptotically vanishing probability of error [2]. The seminal work of Shannon, besides founding the theory of information, has provided the expression for the capacity of the Gaussian channel [3]. On the other hand, the new research field of quantum information science has represented a fertile extension of information theory, with a heritage of open problems, challenges, and physical insights into the foundations and modern applications of quantum physics [4, 5].

Similarly to its classical counterpart, a quantum Gaussian channel is a mathematical model for a communication line associated with a noise process with Gaussian characteristic [6] (see also [7], where this type of maps were considered in view of dynamical invariants [8] for quantum systems with quadratic Hamiltonians). Actually, differently from the classical case, there are several nonequivalent notions of capacities for a quantum channel, depending

on whether the aim of the communication protocol is to transfer classical or quantum information, and on whether entanglement is used to assist the transmission [5]. Physically, the model of quantum Gaussian channel can be applied in the context of continuous variable quantum systems, that is, to the case of bosonic systems, to model linear attenuation and amplification processes. Typical realizations include the quantum electromagnetic field and atomic ensembles.

In the present contribution we consider the problem of evaluating the classical capacity, that is, the maximum rate of reliable transmission of classical information, through a quantum communication line modeled as a quantum Gaussian channel. This problem is formulated as an optimization problem, where a suitable entropic function — the Holevo information — has to be maximized over all possible protocols to encode classical information into an ensemble of quantum states [9]. The main technical difficulty is due to the fact that the Holevo function might be non additive [10], which means that the optimization has to be carried over the whole unbounded set of encoding protocols, including those defined by ensembles of entangled quantum states. While the additivity of the Holevo function has been proved for a class of discrete quantum channels [11], nothing is known in general about continuous ones (to which the quantum Gaussian channels belong to). Then, the problem is dramatically simplified by restricting to the one-shot capacity, i.e., considering only ensembles of separable states. Actually it is further simplified if we also accept the conjecture that the optimal encoding ensemble for a Gaussian channel is solely made of Gaussian states [12]. Here, resorting on these assumptions, we compute lower bounds on the classical capacity of quantum Gaussian channels.

The paper proceeds as follows. In Sec. 2 we briefly introduce the notion and the basic properties of quantum Gaussian channels; in Sec. 3 we considered the problem of estimating the capacity of Gaussian channels and compute lower bounds on the capacity of one-mode Gaussian channels; we end with conclusions in Sec. 4.

2. Gaussian states and channels

The most natural way to introduce the notion of quantum Gaussian channels is via that of Gaussian states. We consider the case of one-mode Gaussian channels, whose input and output quantum systems are described by a canonical pair $\hat{\mathbf{R}} = (\hat{Q}, \hat{P})$, obeying canonical commutation relations $[\hat{Q}, \hat{P}] = i$ (here and in the following we set $\hbar = 1$). A state of the quantum system is described by a density operator $\hat{\rho}$, from which one writes the characteristic function

$$\phi(\mathbf{Z}) = \mathrm{Tr}\left[\hat{\rho}\,\hat{W}(\mathbf{Z})\right] , \tag{1}$$

where $\mathbf{Z} = (X, Y)$ is a two-dimensional real vector, and $\hat{W}(\mathbf{Z}) = e^{i\mathbf{Z}\hat{\mathbf{R}}^{\mathsf{T}}}$ is the Weyl operator. Gaussian states are, by definition, those with Gaussian characteristic function, i.e.,

$$\phi(\mathbf{Z}) = \exp\left(i\mathbf{Z}_0\mathbf{Z}^{\mathsf{T}} - \frac{1}{2}\mathbf{Z}\mathbb{V}\mathbf{Z}^{\mathsf{T}}\right) , \tag{2}$$

where

$$\mathbf{Z}_0 = \mathrm{Tr}\left[\hat{\rho}\,\hat{\mathbf{R}}\right] \tag{3}$$

is the vector of first-moment of the canonical variables $\hat{Q}$, $\hat{P}$, and

$$\mathbb{V} = \mathrm{Tr}\left[\hat{\rho}\begin{pmatrix} \hat{Q}^2 & \frac{\hat{Q}\hat{P}+\hat{P}\hat{Q}}{2} \\ \frac{\hat{Q}\hat{P}+\hat{P}\hat{Q}}{2} & \hat{P}^2 \end{pmatrix}\right] - \mathbf{Z}_0\mathbf{Z}_0^{\mathsf{T}} \tag{4}$$

is the covariance matrix (CM). In addition to be positive definite, $\mathbb{V} > 0$, the uncertainty relation imposes the following constraint on the CM associated with Gaussian states [13]:

$$\det(\mathbb{V}) \geqslant \frac{1}{4}\,. \tag{5}$$

The latter condition can be generalized to the case of multimode Gaussian states, in terms of the symplectic invariants of the CM [13]. Actually, in the case of one-mode Gaussian states, the determinant of the CM is the only symplectic invariant [14]. In the following we will evaluate the capacity of Gaussian channels in terms of entropic functions. We hence recall that the von Neumann entropy, $S[\hat{\rho}] = -\mathrm{Tr}(\hat{\rho}\log_2\hat{\rho})$ (measured in qubits), of a Gaussian state is a function of the symplectic invariants of its CM. For the one-mode Gaussian states we have

$$S[\hat{\rho}] = h\left(\sqrt{\det(\mathbb{V})}\right) = \mathsf{S}\left[\mathbb{V}\right]\,, \tag{6}$$

where we have introduced the function $h(x) = (x + 1/2)\log_2(x+1/2) - (x - 1/2)\log_2(x-1/2)$ [15].

Quantum Gaussian channels are those quantum dynamical maps [5, 16] which transform Gaussian states into Gaussian states [6]. It follows that a Gaussian channel is identified by its action on the CM and on the vector of first moments. The preservation of the Gaussianity of the characteristic function implies that a Gaussian channel is identified by a triplet $(\mathbf{d}, \mathbb{T}, \mathbb{N})$, whose action on the vector of first moment and the CM is as follows:

$$\mathbf{Z}_0 \mapsto \mathbb{T}\mathbf{Z}_0 + \mathbf{d}\,, \tag{7}$$

$$\mathbb{V} \mapsto \mathbb{T}\mathbb{V}\mathbb{T}^{\mathsf{T}} + \mathbb{N}\,, \tag{8}$$

where $\mathbb{N}$ is a 2×2 symmetric, positive semi-definite, matrix, $\mathbb{T}$ is a 2×2 real matrix, and $\mathbf{d}$ is a 2-component real vector. Moreover, the condition of complete positivity on the dynamical map is characterized by the inequality — stated in terms of the symplectic invariants of the matrices $\mathbb{T}$, $\mathbb{N}$ —

$$\det(\mathbb{N}) \geqslant \left[\frac{\det(\mathbb{T}) - 1}{2}\right]^2\,. \tag{9}$$

Among the set of Gaussian channels, a remarkable role is played by Gaussian unitary transformations. These are characterized by triplets of the form $(\mathbf{f}, \mathbb{S}, \mathbb{O})$, where $\mathbf{f}$ is a vector, $\mathbb{S}$ is a symplectic matrix, and $\mathbb{O}$ denotes the null matrix. The composition, from the left and from the right, of the Gaussian channel with the Gaussian unitaries $(\mathbf{f}_A, \mathbb{S}_A, \mathbb{O})$, $(\mathbf{f}_B, \mathbb{S}_B, \mathbb{O})$, transforms the associated triplet according to

$$\mathbf{d} \mapsto \mathbb{S}_B(\mathbb{T}\mathbf{f}_A + \mathbf{d}) + \mathbf{f}_B\,, \tag{10}$$

$$\mathbb{T} \mapsto \mathbb{S}_B\mathbb{T}\mathbb{S}_A\,, \tag{11}$$

$$\mathbb{N} \mapsto \mathbb{S}_B\mathbb{N}\mathbb{S}_B^{\mathsf{T}}\,. \tag{12}$$

Class	$\mathbb{T}_c$	$\mathbb{N}_c$	Range of $\tau = \det(\mathbb{T})$
$\mathcal{A}_1$	$\mathbb{O}$	$(\bar{n}+1/2)\mathbb{I}$	$\{0\}$
$\mathcal{A}_2$	$\frac{\mathbb{I}+\mathbb{Z}}{2}$	$(\bar{n}+1/2)\mathbb{I}$	$\{0\}$
$\mathcal{B}_1$	$\mathbb{I}$	$\frac{\mathbb{I}+\mathbb{Z}}{2}$	$\{1\}$
$\mathcal{B}_2$	$\mathbb{I}$	$\bar{n}\mathbb{I}$	$\{1\}$
$\mathcal{C}$ (Att)	$\sqrt{\tau}\mathbb{I}$	$(1-\tau)(\bar{n}+1/2)\mathbb{I}$	$(0,1)$
$\mathcal{C}$ (Amp)	$\sqrt{\tau}\mathbb{I}$	$(\tau-1)(\bar{n}+1/2)\mathbb{I}$	$(1,\infty)$
$\mathcal{D}$	$\sqrt{-\tau}\mathbb{Z}$	$(1-\tau)(\bar{n}+1/2)\mathbb{I}$	$(-\infty,0)$

Table 1. Equivalence classes of one-mode Gaussian channels. Each class is identified by the matrices $\mathbb{T}_c$, $\mathbb{N}_c$. $\mathbb{I}$ and $\mathbb{O}$ respectively denote the identity and null matrices, $\mathbb{Z} = \mathrm{diag}(1,-1)$.

From the point of view of information theory, the composition from the left and from the right of a Gaussian channel with Gaussian unitaries corresponds to unitary pre-processing and post-processing of the channel. The Gaussian channels can be classified according to equivalence up to Gaussian unitary transformations: Two Gaussian channels are said to be equivalent if there exist Gaussian unitary pre-processing and post-processing mapping one channel into the other. For one-mode Gaussian channels, different equivalence classes can be identified [17], which are summarized in Table 1. For each class, one can pick up a representative Gaussian channel associated with a triplet in the canonical form $(\mathbf{0}, \mathbb{T}_c, \mathbb{N}_c)$, where $\mathbf{0} = \mathbb{S}_B(\mathbb{T}\mathbf{f}_A + \mathbf{d}) + \mathbf{f}_B$, $\mathbb{T}_c = \mathbb{S}_B\mathbb{T}\mathbb{S}_A$, $\mathbb{N}_c = \mathbb{S}_B\mathbb{N}\mathbb{S}_B^{\mathsf{T}}$, and both matrices $\mathbb{T}_c$, $\mathbb{N}_c$ are diagonal (see Table 1 for details). Besides the rank-deficient classes $\mathcal{A}_2$, $\mathcal{B}_1$, each class is identified by the range of the parameter $\tau := \det(\mathbb{T})$. Within each class, a canonical form is characterized by the values of τ and of the non-negative parameter $\bar{n}$. The canonical form can be physically represented in terms of two-mode Gaussian unitary transformations, where an ancillary mode is introduced in a thermal state [17]: The class $\mathcal{C}$ describes linear attenuation (for $\tau < 1$) and amplification (for $\tau > 1$) processes, which are commonly modeled in terms of beam-splitters and linear amplifiers; The class $\mathcal{D}$ is the conjugate channel of the linear amplifier; Class $\mathcal{A}_1$ models the erasure process, and can be represented as the limit of the attenuating channel for $\tau \to 0$. Class $\mathcal{B}_2$ models the action of classical noise, its Stinespring dilation can be obtained with the help of two ancillary modes [**?**]. Finally, the classes $\mathcal{A}_2$, $\mathcal{B}_1$ present special features, they are characterized by rank deficient canonical matrices and can be obtained as singular limits of the previous cases.

3. Classical capacity of one-mode Gaussian channels

An encoding procedure is identified by a map $\mathbf{x} \to \hat{\rho}_{\mathbf{x}}$, which assigns a quantum state to each value of a stochastic variable, distributed according to a probability density $p_{\mathbf{x}}$.

If the quantum states are subjected to the action of a quantum channel $\hat{\mathcal{E}}$, the receiver has to extract information from an ensemble composed of the states $\hat{\mathcal{E}}(\hat{\rho}_{\mathbf{x}})$. According to [9], the maximum information, measured in bits, that can be extracted from the enseble is given by

the Holevo information:

$$\chi = S\left[\int d\mathbf{x} p_{\mathbf{x}} \hat{\mathcal{E}}(\hat{\rho}_{\mathbf{x}})\right] - \int d\mathbf{x} p_{\mathbf{x}} S\left[\hat{\mathcal{E}}(\hat{\rho}_{\mathbf{x}})\right] , \tag{13}$$

where S is the von Neumann entropy.

The capacity of a quantum Gaussian channel $\hat{\mathcal{E}}$, associated with the triplet $(\mathbf{d}, \mathbb{T}, \mathbb{N})$, is the supremum of the quantity (13) over all possible encoding procedures. Among them, we consider Gaussian encodings, in which the value $\mathbf{x}$ of the stochastic variable is encoded in Gaussian states with CM $\mathbb{V}$ and first-moment $\mathbf{d} = \mathbf{x}$. Furthermore, we assume a Gaussian form for the distribution $p_{\mathbf{x}}$, with zero mean and CM $\mathbb{M}$, which is symmetric and positive semi-definite. In these settings, we have that

$$S\left[\hat{\mathcal{E}}(\hat{\rho}_{\mathbf{x}})\right] = \mathsf{S}\left[\mathbb{T}\mathbb{V}\mathbb{T}^{\mathsf{T}} + \mathbb{N}\right] , \tag{14}$$

$$S\left[\int d\mathbf{x} p_{\mathbf{x}} \hat{\mathcal{E}}(\hat{\rho}_{\mathbf{x}})\right] = \mathsf{S}\left[\mathbb{T}(\mathbb{V}+\mathbb{M})\mathbb{T}^{\mathsf{T}} + \mathbb{N}\right] . \tag{15}$$

However, the maximization of the Holevo information, even when restricted to Gaussian encodings, can lead to an infinite value of the channel capacity. This is due to the fact that the relevant Hilbert space is infinite-dimensional (and, hence, it can in principle store an infinite amount of classical information), and that the manifold of Gaussian states is not compact. An effective cutoff in the Hilbert space can be introduced by imposing a constraint on the average mean energy involved in the encoding process ‡:

$$\frac{1}{2}\mathrm{Tr}(\mathbb{V}+\mathbb{M}) \leqslant E . \tag{16}$$

The optimization over Gaussian encodings furnishes a lower bound on the capacity of the Gaussian channel:

$$\underline{C} = \max_{\mathbb{V},\mathbb{M}} \left\{\mathsf{S}\left[\mathbb{T}(\mathbb{V}+\mathbb{M})\mathbb{T}^{\mathsf{T}} + \mathbb{N}\right] - \mathsf{S}\left[\mathbb{T}\mathbb{V}\mathbb{T}^{\mathsf{T}} + \mathbb{N}\right]\right\} , \tag{17}$$

where the maximum is over the CMs $\mathbb{V}$, $\mathbb{M}$ subjected to the uncertainty relation (5) and to the energy constrain (16). In the following the maximization problem will be solved by using the Karush-Kuhn-Tucker (KKT) method [2], which generalizes the Lagrange method to the case of constraints expressed by inequalities.

Before proceeding with the evaluation of the lower bounds on the capacity for the Gaussian channels belonging to different equivalence classes, we remark that the von Neumann entropy is invariant under symplectic transformations, that is,

$$\mathsf{S}\left[\mathbb{T}\mathbb{V}\mathbb{T}^{\mathsf{T}} + \mathbb{N}\right] = \mathsf{S}\left[\mathbb{S}_B\mathbb{T}\mathbb{V}\mathbb{T}^{\mathsf{T}}\mathbb{S}_B^{\mathsf{T}} + \mathbb{S}_B\mathbb{N}\mathbb{S}_B^{\mathsf{T}}\right] , \tag{18}$$

for any symplectic matrix $\mathbb{S}_B$, which is interpreted as a post-processing Gaussian unitary transformation. It follows that the function (17) can be rewritten as follows

$$\underline{C} = \max_{\mathbb{V},\mathbb{M}} \left\{\mathsf{S}\left[\mathbb{T}_c\mathbb{S}_A(\mathbb{V}+\mathbb{M})\mathbb{S}_A^{\mathsf{T}}\mathbb{T}_c^{\mathsf{T}} + \mathbb{N}_c\right] - \mathsf{S}\left[\mathbb{T}_c\mathbb{S}_A\mathbb{V}\mathbb{S}_A^{\mathsf{T}}\mathbb{T}_c^{\mathsf{T}} + \mathbb{N}_c\right]\right\} . \tag{19}$$

‡ Clearly, other choices are possible, e.g., to constraint the output energy. However, since the Holevo information (13) is obtained by optimizing over all the possible measurement on the channel output, we do not assume any constraint on the output of the channel.

This expression is written in terms of the canonical matrices and the pre-processing symplectic matrix $\mathbb{S}_A$. The latter can be further decomposed according to Euler decomposition:

$$\mathbb{S}_A = \mathbb{R}\mathbb{D}\mathbb{R}' \,, \tag{20}$$

where $\mathbb{R}$, $\mathbb{R}'$ are orthogonal matrices and $\mathbb{D} = \mathrm{diag}(r^{1/2}, r^{-1/2})$. Since the orthogonal matrix $\mathbb{R}'$ preserves the constrain (16), it can be eliminated by redefining the CMs $\mathbb{V}$, $\mathbb{M}$, yielding

$$\underline{C} = \max_{\mathbb{V},\mathbb{M}} \left\{ \mathsf{S}\left[\mathbb{T}_c\mathbb{R}\mathbb{D}(\mathbb{V}+\mathbb{M})(\mathbb{T}_c\mathbb{R}\mathbb{D})^{\mathsf{T}} + \mathbb{N}_c\right] - \mathsf{S}\left[\mathbb{T}_c\mathbb{R}\mathbb{D}\mathbb{V}(\mathbb{T}_c\mathbb{R}\mathbb{D})^{\mathsf{T}} + \mathbb{N}_c\right] \right\} . \tag{21}$$

3.1. Class $\mathcal{A}_1$: erasure channel

Since $\mathbb{T}_c = \mathbb{O}$, for this class of channels we trivially have

$$\underline{C} = \max_{\mathbb{V},\mathbb{M}} \left\{ \mathsf{S}\left[\mathbb{N}_c\right] - \mathsf{S}\left[\mathbb{N}_c\right] \right\} = 0 \,. \tag{22}$$

Actually, by allowing non-Gaussian encoding we can similarly prove that the channel capacity itself vanishes.

3.2. Class $\mathcal{A}_2$

For the channels belonging to this class, $\mathbb{T}_c = (\mathbb{I}+\mathbb{Z})/2 = \mathrm{diag}(1,0)$ and $\mathbb{N}_c = (\bar{n}+1/2)\mathbb{I}$. Since the matrix $\mathbb{T}_c\mathbb{R}\mathbb{D}$ appearing in Eq. (21) has rank one, its singular value decomposition reads

$$\mathbb{T}_c\mathbb{R}\mathbb{D} = \sqrt{t}\,\mathbb{R}'\mathbb{T}_c\mathbb{R}'' \,, \tag{23}$$

where $\mathbb{R}'$, $\mathbb{R}''$ are orthogonal matrices, and $\sqrt{t}$ is the (non-vanishing) singular value of $\mathbb{T}_c\mathbb{R}\mathbb{D}$. Absorbing the orthogonal matrix $\mathbb{R}''$ into the definition of the CMs $\mathbb{V}$ and $\mathbb{M}$, we obtain the following expression for the lower bound:

$$\underline{C} = \max_{\mathbb{V},\mathbb{M}} \left\{ \mathsf{S}\left[t\mathbb{T}_c(\mathbb{V}+\mathbb{M})\mathbb{T}_c + \mathbb{N}_c\right] - \mathsf{S}\left[t\mathbb{T}_c\mathbb{V}\mathbb{T}_c + \mathbb{N}_c\right] \right\} \,. \tag{24}$$

By introducing the parameterization $\mathbb{V} = \mathrm{diag}(s^{-1}/2, s/2)$, Eq. (24) can be rewritten as the maximum of a function of s:

$$\underline{C} = \max_{s\in[s_-,s_+]} \left\{ h(\bar{\nu}_{\mathcal{A}_2}) - h(\nu_{\mathcal{A}_2}) \right\} \,, \tag{25}$$

where

$$\bar{\nu}_{\mathcal{A}_2} = \sqrt{\left[t\left(2E+1-\frac{s}{2}\right) + \bar{n} + \frac{1}{2}\right]\left(\bar{n}+\frac{1}{2}\right)} \,, \tag{26}$$

$$\nu_{\mathcal{A}_2} = \sqrt{\left(t\frac{s^{-1}}{2} + \bar{n} + \frac{1}{2}\right)\left(\bar{n}+\frac{1}{2}\right)} \,, \tag{27}$$

and $s_\pm = 2E+1 \pm \sqrt{(2E+1)^2-1}$.

3.3. Class $\mathcal{B}_1$

For the elements belonging to this class, $\mathbb{T}_c = \mathbb{I}$ and $\mathbb{N}_c = (\mathbb{I}+\mathbb{Z})/2 = \mathrm{diag}(1,0)$, Eq. (21) reads

$$\underline{C} = \max_{\mathbb{V},\mathbb{M}} \left\{ \mathsf{S}\left[\mathbb{R}\mathbb{D}(\mathbb{V}+\mathbb{M})(\mathbb{R}\mathbb{D})^{\mathsf{T}} + \mathbb{N}_c\right] - \mathsf{S}\left[\mathbb{T}_c\mathbb{R}\mathbb{D}\mathbb{V}(\mathbb{T}_c\mathbb{R}\mathbb{D})^{\mathsf{T}} + \mathbb{N}_c\right] \right\} \quad (28)$$

$$= \max_{\mathbb{V},\mathbb{M}} \left\{ \mathsf{S}\left[\mathbb{V}+\mathbb{M}+\mathbb{D}^{-1}\mathbb{R}^{\mathsf{T}}\mathbb{N}_c\mathbb{R}\mathbb{D}^{-1}\right] - \mathsf{S}\left[\mathbb{V}+\mathbb{D}^{-1}\mathbb{R}^{\mathsf{T}}\mathbb{N}_c\mathbb{R}^{\mathsf{T}}\mathbb{D}^{-1}\right] \right\} . \quad (29)$$

The matrix $\mathbb{D}^{-1}\mathbb{R}^{\mathsf{T}}\mathbb{N}_c\mathbb{R}\mathbb{D}^{-1}$ is symmetric with rank one, hence we can write it as follows:

$$\mathbb{D}^{-1}\mathbb{R}^{\mathsf{T}}\mathbb{N}_c\mathbb{R}\mathbb{D}^{-1} = n\mathbb{R}'\mathbb{N}_c\mathbb{R}'^{\mathsf{T}} , \quad (30)$$

where n is its (non-vanishing) eigenvalue, and $\mathbb{R}'$ is orthogonal. Absorbing $\mathbb{R}'$ into the definition of the CMs $\mathbb{V}$, $\mathbb{M}$, we obtain

$$\underline{C} = \max_{\mathbb{V},\mathbb{M}} \left\{ \mathsf{S}\left[\mathbb{V}+\mathbb{M}+n\mathbb{N}_c\right] - \mathsf{S}\left[\mathbb{V}+n\mathbb{N}_c\right] \right\} . \quad (31)$$

Using the parameterization $\mathbb{V} = \mathrm{diag}(s^{-1}/2, s/2)$, Eq. (31) yields

$$\underline{C} = \max_{s\in[s_-,s_+]} \left\{ h(\bar{\nu}_{\mathcal{B}_1}) - h(\nu_{\mathcal{B}_1}) \right\} , \quad (32)$$

where

$$\bar{\nu}_{\mathcal{B}_1} = \sqrt{\left(\frac{s^{-1}}{2}+n\right)\left(2E+1-\frac{s^{-1}}{2}\right)} , \quad (33)$$

$$\nu_{\mathcal{B}_1} = \sqrt{\frac{1}{4}+n\frac{s}{2}} . \quad (34)$$

3.4. Class $\mathcal{C}$: attenuating and amplifying channels

For this class we have $\mathbb{T}_c = \sqrt{\tau}\mathbb{I}$, $\mathbb{N}_c = |1-\tau|(\bar{n}+1/2)\mathbb{I}$, where $\tau \in (0,1)$ for the attenuating channels, and $\tau \in (1,\infty)$ for the amplifying ones. Equation (21) simplifies as follows

$$\underline{C} = \max_{\mathbb{V},\mathbb{M}} \left\{ \mathsf{S}\left[\tau\mathbb{D}(\mathbb{V}+\mathbb{M})\mathbb{D}+\mathbb{N}_c\right] - \mathsf{S}\left[\tau\mathbb{D}\mathbb{V}\mathbb{D}+\mathbb{N}_c\right] \right\} . \quad (35)$$

First of all, we notice that the matrices $\mathbb{D}$, $\mathbb{N}_c$ in Eq. (35) are all diagonal, hence the maximum under the constraint (16) is reached in correspondence to diagonal CMs $\mathbb{V}$, $\mathbb{M}$. The constrained optimization yields the analytic solution

$$\underline{C} = h\left[\tau\left(E+\frac{1}{2}\right)+|1-\tau|\left(\bar{n}+\frac{1}{2}\right)\frac{r+r^{-1}}{2}\right] - h\left[\frac{\tau}{2}+|1-\tau|\left(\bar{n}+\frac{1}{2}\right)\right] . \quad (36)$$

The latter is obtained in the region of parameters for which

$$\left(E+\frac{1}{2}\right) \pm \frac{|1-\tau|}{\tau}\frac{r-r^{-1}}{2}\left(\bar{n}+\frac{1}{2}\right) - \frac{r^{\mp 1}}{2} \geqslant 0 . \quad (37)$$

Outside this region the KKT method does not lead to an analytical expression for the capacity lower bound, which can however still be evaluated numerically by solving a transcendental equation. The existence of qualitative differences in the solution of the optimization problems according to the range of the parameters has been put forward and analyzed in details in [19] (see also [20] and [21]).

3.5. Class $\mathcal{B}_2$: additive noise channels

For the channels belonging to the equivalence class of the additive noise channel, $\mathbb{T}_c = \mathbb{I}$, $\mathbb{N}_c = \bar{n}\mathbb{I}$, and Eq. (21) reads

$$\underline{C} = \max_{\mathbb{V},\mathbb{M}} \left\{ \mathsf{S}\left[\mathbb{D}(\mathbb{V}+\mathbb{M})\mathbb{D}^\mathsf{T} + \mathbb{N}_c\right] - \mathsf{S}\left[\mathbb{D}\mathbb{V}\mathbb{D}^\mathsf{T} + \mathbb{N}_c\right] \right\} , \tag{38}$$

This expression can be obtained from (35) by replacing τ with 1, and $|1-\tau|(\bar{n}+1/2)$ with $\bar{n}$. Under the conditions

$$E + \frac{1}{2} \pm \bar{n}\frac{r - r^{-1}}{2} - \frac{r^{\mp 1}}{2} \geqslant 0 \,, \tag{39}$$

the optimization yields the following analytical expression

$$\underline{C} = h\left(E + \frac{1}{2} + \bar{n}\frac{r + r^{-1}}{2}\right) - h\left(\bar{n} + \frac{1}{2}\right) . \tag{40}$$

Outside the region of parameters defined by Eq. (39) the KKT method does not lead to a closed form for $\underline{C}$, see also [18].

3.6. Class $\mathcal{D}$: Conjugate of the amplifying channel

For the channels belonging to this class, $\mathbb{T}_c = \sqrt{-\tau}\mathbb{Z}$, and $\mathbb{N}_c = (1-\tau)(\bar{n}+1/2)\mathbb{I}$, where $\tau \in (-\infty, 0)$ and $\mathbb{Z} = \mathrm{diag}(1,-1)$, yielding the following form for Eq. (21):

$$\underline{C} = \max_{\mathbb{V},\mathbb{M}} \left\{ \mathsf{S}\left[|\tau|\mathbb{Z}\mathbb{R}\mathbb{D}(\mathbb{V}+\mathbb{M})(\mathbb{R}\mathbb{D})^\mathsf{T}\mathbb{Z} + \mathbb{N}_c\right] - \mathsf{S}\left[|\tau|\mathbb{Z}\mathbb{R}\mathbb{D}\mathbb{V}(\mathbb{R}\mathbb{D})^\mathsf{T}\mathbb{Z} + \mathbb{N}_c\right] \right\} . \tag{41}$$

Since the von Neumann entropy S is a function of the determinant of its argument, we can write

$$\begin{aligned} \underline{C} &= \max_{\mathbb{V},\mathbb{M}} \left\{ \mathsf{S}\left[|\tau|\mathbb{R}\mathbb{D}(\mathbb{V}+\mathbb{M})(\mathbb{R}\mathbb{D})^\mathsf{T} + \mathbb{Z}\mathbb{N}_c\mathbb{Z}\right] - \mathsf{S}\left[|\tau|\mathbb{R}\mathbb{D}\mathbb{V}(\mathbb{R}\mathbb{D})^\mathsf{T} + \mathbb{Z}\mathbb{N}_c\mathbb{Z}\right] \right\} \\ &= \max_{\mathbb{V},\mathbb{M}} \left\{ \mathsf{S}\left[|\tau|\mathbb{D}(\mathbb{V}+\mathbb{M})\mathbb{D}^\mathsf{T} + \mathbb{N}_c\right] - \mathsf{S}\left[|\tau|\mathbb{D}\mathbb{V}\mathbb{D}^\mathsf{T} + \mathbb{N}_c\right] \right\} . \end{aligned} \tag{42}$$

The latter expression is identical to Eq. (35), upon replacing τ with $|\tau|$, and $|1-\tau|$ with $1+|\tau|$. Hence the solution of the maximization problem is obtained as for the class $\mathcal{C}$, yielding

$$\begin{aligned} \underline{C} &= h\left[|\tau|\left(E + \frac{1}{2}\right) + (1+|\tau|)\left(\bar{n} + \frac{1}{2}\right)\frac{r + r^{-1}}{2}\right] \\ &- h\left[\frac{|\tau|}{2} + (1+|\tau|)\left(\bar{n} + \frac{1}{2}\right)\right] , \end{aligned} \tag{43}$$

under the conditions

$$\left(E + \frac{1}{2}\right) \pm \frac{(1+|\tau|)}{|\tau|}\frac{r - r^{-1}}{2}\left(\bar{n} + \frac{1}{2}\right) - \frac{r^{\mp 1}}{2} \geqslant 0 \,. \tag{44}$$

4. Conclusions

We have considered the whole family of one-mode quantum Gaussian channels, which can be classified into equivalence classes, and, by using Gaussian encoding, we have evaluated lower bounds on the classical capacity under input energy constraint.

For small enough values of the energy the optimal encodings can be evaluated numerically, while a closed form for the lower bound has been obtained when the allowed energy is above a certain threshold. The presence of two different regimes for the optimization problem has been studied in details in [19] (see also [18, 20, 21]). The region of parameters allowing an analytical expression for the lower bound on the capacity includes the limit $E \to \infty$. Using the fact that $h(x) \simeq \log_2 x + \log_2 e$ for $x \to \infty$, we have the following high-energy limits of the capacity lower bounds:

$$\underline{C} \simeq \begin{cases} 0 & \text{class } \mathcal{A}_1 \\ \log_2 e + \log_2(E) & \text{class } \mathcal{B}_1 \\ \log_2 e + \log_2(E) - h(\bar{n} + 1/2) & \text{class } \mathcal{B}_2 \\ \log_2 e + \log_2(|\tau|E) - h[|\tau|/2 + |1-\tau|(\bar{n}+1/2)] & \text{classes } \mathcal{C}, \mathcal{D} \end{cases}$$

The Holevo function (21) assumes different forms, depending on the equivalence class which the Gaussian channel belongs to. However, interestingly enough, the value of its maximum is a function of the channel itself and is not uniquely determined by the property of belonging to a given class.

This is reminiscent of other problems in quantum information science for which the orbits of the action of the relevant unitary group do not provide a complete characterization of the theoretical framework [22].

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